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Massera's Convergence Theorem for Periodic Nonlinear Differential Equations

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1. INTRODUCTION

Ordinary differential equations which are periodic in the independent variable can arise from many problems in mechanics and circuit theory. For these applications it is interesting to know about the existence of periodic solutions and particularly about those which are stable. Powerful tools for proving the existence of periodic solutions are provided by fixed point theory and degree theory and many papers have been written about this. However, relatively few papers have been written about the existence of stable periodic solutions (see [12, p. 227]). When a periodic solution has been calculated explicitly its stability can normally be determined by computing the Floquet multipliers of its perturbational equation and verifying that these all have modulus less than 1. This always involves considerable effort and is possible only when the periodic solution is known with some accuracy. However, the qualitative theory of differential equation aims to predict the existence of a stable periodic solution without requiring it to be computed explicitly. It is possible to do this for certain special classes of differential equations by methods described in [3, 5, 6, 7]. One aim of the present paper is to provide a new class of equations for which it is practicable to predict the existence of a stable periodic solution.

This paper was inspired by results of Massera and Pliss concerning scalar differential equations of the first order which are σ-periodic in the independent variable. Massera [9] showed that if such an equation has a solution $x(t)$ which is bounded in some interval $t_0 \leqslant t < \infty$ then $x(t)$ converges to a σ-periodic solution as $t \to +\infty$. For the same equations Pliss [10, Sect. 9] proved results on stability and on the number of periodic solutions of analytic equations. It is well known that these theorems are not valid for all differential equations in $\mathbb{R}^n$ when $n > 1$. In Sections 2, 3 of the present paper we obtain analogues of Massera's convergence theorem

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and the associated results of Pliss for a special class of differential equations in $\mathbb{R}^n$. The stability results in Section 3 are unusual because they predict the presence of a stable periodic solution in a region which may also contain many unstable solutions.

Massera [9] used the convergence theorem to motivate his more famous result concerning σ-periodic differential equations in $\mathbb{R}^2$ for which each solution has an interval of existence of the form (θ, ∞). He showed that if such an equation has a solution which is bounded in the future then it also has a σ-periodic solution. This is a delicate result because equations satisfying these hypotheses may also have recurrent solutions which are not periodic. In Section 4 we obtain an analogue of Massera's 2-dimensional theorem for a special class of differential equations in $\mathbb{R}^n$. This is proved by making a detailed study of a class of solutions in $\mathbb{R}^n$ called the amenable solutions. As a by-product of this work on amenable solutions we also obtain in Section 4 a refinement of a result of Cartwright [2] concerning the frequency spectrum of uniformly almost periodic solutions.

The classes of differential equations in $\mathbb{R}^n$ to which our theorems apply are specified by certain hypotheses (H3), (H4) described in Section 2. Their verification is discussed in Sections 5, 6. For generalised feedback control equations these hypotheses are reduced in Section 5 to an inequality which is analogous to the frequency domain criterion used by control engineers for stability problems. Its geometrical interpretation as a circle criterion is described in Section 6 for the special case when the nonlinearities of the differential equation are one-dimensional. It is shown how this enables the analogues of Massera's theorems to be applied easily to a considerable class of nonlinear differential equations of higher order.

2. Massera's Convergence Theorem

Throughout this paper x^* and $|x|$ denote the transpose and euclidean norm of any column vector x in $\mathbb{R}^n$. We consider the vector differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

in which $f(t, x)$ is a continuous function from $\mathbb{R} \times S$ into $\mathbb{R}^n$ for some open subset S of $\mathbb{R}^n$. To ensure that solutions in S are uniquely determined by their initial values and vary continuously with them we assume the following:

(H1) *$f(t, x)$ satisfies a local Lipschitz condition in $\mathbb{R} \times S$.*

Except in parts of Sections 3, 4 the following hypothesis is also assumed:

(H2) *there exists a constant* $\sigma > 0$ *such that* $f(t+\sigma, x) = f(t, x)$ *in* $\mathbb{R} \times S$.

If S_0 is a compact subset of S then (H1) implies a global Lipschitz condition in the compact set $[0, \sigma] \times S_0$. That is, there exists a constant $\gamma(S_0)$ such that

$$|f(t, x_1) - f(t, x_2)| \leqslant \gamma(S_0)|x_1 - x_2| \qquad \text{for } x_1, x_2 \in S_0 \tag{2}$$

and $0 \leqslant t \leqslant \sigma$. This restriction $0 \leqslant t \leqslant \sigma$ can be ignored when $f(t, x)$ satisfies (H2). If $x(t)$, $y(t)$ are solutions of (1) such that $x(t), y(t) \in S_0$ for $\theta \leqslant t \leqslant \tau$ then (2) gives

$$\begin{aligned} |x(\theta) - y(\theta)| \exp[-\gamma(S_0)(\tau - \theta)] &\leqslant |x(\tau) - y(\tau)| \\ &\leqslant |x(\theta) - y(\theta)| \exp[\gamma(S_0)(\tau - \theta)]. \end{aligned} \tag{3}$$

For the special case of scalar equations which satisfy (H1), (H2) with $S = \mathbb{R}$, Massera [9] showed that any solution $y(t)$ which is bounded in an interval (t_0, ∞) must converge to a σ-periodic solution $u(t)$ as $t \to +\infty$. This result can only be extended to higher-dimensional equations by adding further hypotheses to it. For equations satisfying (H1), (H2) with $S = \mathbb{R}^n$ and $n \geqslant 1$, Sell [13, p. 151] showed that any bounded solution $y(t)$ which is also uniformly asymptotically stable must converge to a periodic solution $u(t)$ as $t \to +\infty$. In practice the asymptotic stability of $y(t)$ could be verified by using a theorem of Demidovich [5]. For the special case of autonomous equations the periodic solution $u(t)$ reduces to a constant solution in the above theorems of Massera and Sell. For this special case, a more delicate version discussed by Cronin [3, p. 250; 4, p. 238] states that if $y(t)$ is a bounded phase asymptotically stable solution then the set of ω-limit points of $y(t)$ is the orbit of a phase asymptotically stable periodic solution. This result is closely related to the Poincaré–Bendixson theorem for plane autonomous equations. Cronin [3] also gave sufficient conditions for $y(t)$ to be phase asymptotically stable.

The present paper is concerned mainly with nonautonomous equations. In this section a new higher-dimensional analogue of Massera's theorem is obtained by adding to it hypothesis which are different to those of Sell and Cronin. The main hypothesis is as follows:

(H3) *there exist constants* $\lambda \geqslant 0$, $\varepsilon > 0$ *and a constant real symmetric* $n \times n$ *matrix* P *such that for all real* t,

$$\begin{aligned} (x_1 - x_2)^* P[f(t, x_1) - f(t, x_2) + \lambda(x_1 - x_2)] \\ \leqslant -\varepsilon |x_1 - x_2|^2 \qquad \textit{for } x_1, x_2 \in S. \end{aligned} \tag{4}$$

If $V(x)=x^*Px$ and $x(t)$, $y(t)$ are solutions of (1), this gives

$$\begin{aligned}&\frac{d}{dt}[e^{2\lambda t}V(x(t)-y(t))]\\&\qquad=2e^{2\lambda t}(x-y)^*P[f(t,x)-f(t,y)+\lambda(x-y)],\\&\qquad\leqslant -2\varepsilon|x(t)-y(t)|^2\,e^{2\lambda t},\end{aligned}\tag{5}$$

for all t such that $x(t), y(t)\in S$. If $x(t), y(t)\in S$ for $\theta\leqslant t\leqslant\tau$ this shows that $e^{2\lambda t}V(x(t)-y(t))$ is monotonic decreasing in $[\theta,\tau]$ and strictly decreasing when the solutions $x(t)$, $y(t)$ are distinct. By integrating (5) over the interval $[\theta,\tau]$ we also get

$$\begin{aligned}&e^{2\lambda\theta}V(x(\theta)-y(\theta))-e^{2\lambda\tau}V(x(\tau)-y(\tau))\\&\qquad\geqslant 2\varepsilon\int_\theta^\tau e^{2\lambda t}|x(t)-y(t)|^2\,dt.\end{aligned}\tag{6}$$

To show how the present paper is related to earlier work we prove first the following result:

THEOREM 1. *Suppose that* (1) *satisfies* (H1), (H2), (H3) *with* $\lambda=0$. *If* (1) *has a solution* $y(t)$ *which remains in a compact subset* S_0 *of* S *throughout* $t_0\leqslant t<\infty$ *then* (1) *has a* σ*-periodic solution* $u(t)$ *such that* $y(t)-u(t)\to 0$ *as* $t\to+\infty$. *Furthermore,* $u(t)$ *is the only periodic solution of* (1) *which lies in* S *for all* t.

Since $\lambda=0$, it is clear from (5) that the stability of the solution $u(t)$ can be determined by Lyapunov's second method using $V(x)$ as Lyapunov function. If $V(x)$ is positive definite then $u(t)$ is asymptotically stable. It is unstable if the matrix P has any negative eigenvalues. In the special case when $V(x)$ is positive definite, Theorem 1 is essentially the same as the convergence theorem of Demidovich [5].

Proof of Theorem 1. If solutions $x(t)$, $y(t)$ of (1) both lie in the compact set S_0 throughout $[t_0,\infty)$ then there exists a constant $K\geqslant|V(x(t)-y(t))|$ for all $t\geqslant t_0$. Since $\lambda=0$, (6) gives $K\geqslant\varepsilon\int_\theta^\tau|x(t)-y(t)|^2\,dt$ for $t_0\leqslant\theta\leqslant\tau$. Hence $\int_\theta^\infty|x(t)-y(t)|^2\,dt$ converges and the left-hand side of (3) gives

$$\begin{aligned}\int_\theta^\infty|x(t)-y(t)|^2\,dt&\geqslant|x(\theta)-y(\theta)|^2\int_\theta^\infty\exp[-2\gamma(S_0)(t-\theta)]\,dt\\&=[2\gamma(S_0)]^{-1}|x(\theta)-y(\theta)|^2,\end{aligned}$$

for all $\theta\geqslant t_0$. This proves that

$$x(\theta)-y(\theta)\to 0\qquad\text{as }\theta\to+\infty.\tag{7}$$

In particular, (7) holds when $x(t)=y(t+\sigma)$ because the solution $y(t+\sigma)\in S_0$ for all $t\geqslant t_0$. Since S_0 is compact, Weierstrass's theorem gives

c in S_0 and a strictly increasing sequence of positive integers $m(1)$, $m(2)$, $m(3)$,..., such that $y(t_0+m(\nu)\sigma) \to c$ as $\nu \to \infty$. If the solution $u(t)$ of (1) having $u(t_0)=c$ lies in S throughout the interval $t_0 \leqslant t < \alpha$ then $y(t+m(\nu)\sigma) \to u(t)$ pointwise in $[t_0, \alpha)$ as $\nu \to \infty$. This solution $u(t)$ cannot leave the compact set S_0 in $[t_0, \alpha)$ because, if it did so, then some neighbouring solution $y(t+m(\nu)\sigma)$ would also leave S_0, contradicting $y(t) \in S_0$ for $t \geqslant t_0$. Hence, $u(t) \in S_0$ throughout $[t_0, \infty)$ and $y(t+m(\nu)\sigma) \to u(t)$ pointwise in $[t_0, \infty)$ as $\nu \to \infty$. Then

$$u(t_0+\sigma) = \lim y(t_0+\sigma+m(\nu)\sigma) = \lim y(t_0+m(\nu)\sigma) = c = u(t_0),$$

because (7) gives $y(t+\sigma)-y(t) \to 0$ as $t \to +\infty$. This proves that $u(t)$ is a σ-periodic solution such that $u(t) \in S_0$ for all t. Then (7) gives $u(t)-y(t) \to 0$ as $t \to +\infty$.

If $\hat{u}(t)$ is another periodic solution such that $\hat{u}(t) \in S$ for all t, we can choose a larger compact set $S_0 \subset S$ such that $u(t)$, $\hat{u}(t) \in S_0$ for all t. Then (7) gives $u(t)-\hat{u}(t) \to 0$ as $t \to +\infty$. This can only happen if the periodic solutions $u(t)$, $\hat{u}(t)$ coincide. Hence, $u(t)$ is the only periodic solution of (1) in S. This establishes Theorem 1.

Since it is clear from (4) that the symmetric matrix P is non-singular, there exists an integer j satisfying the following:

(H4) *P has j negative eigenvalues and $n-j$ positive eigenvalues.*

It will be shown in Section 5 that for each integer j with $0 \leqslant j \leqslant n$ there exists a class of differential equations for which both (H3) and (H4) hold. When P satisfies (H4) there exists an invertible real $n \times n$ matrix M such that $M^*PM = \operatorname{diag}(-I_j, I_{n-j})$, where I_r denotes unit $r \times r$ matrix. The quadratic form $V(x) = x^*Px$ is therefore reduced to the canonical form $V(x) = Y^2 - X^2$ by the linear substitution $x = M\operatorname{col}(X, Y)$ in which $X \in \mathbb{R}^j$ and $Y \in \mathbb{R}^{n-j}$. Let $\Pi: \mathbb{R}^n \to \mathbb{R}^j$ be the linear mapping defined by $\Pi x = X$ for all x in $\mathbb{R}^n$. Since $|M^{-1}x|^2 = X^2 + Y^2$ we have

$$V(x) + 2|\Pi x|^2 = |M^{-1}x|^2 \geqslant |\Pi x|^2 \qquad \text{for } x \in \mathbb{R}^n. \tag{8}$$

The main result of this section is as follows:

THEOREM 2. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4) *with $\lambda > 0$ and $j = 1$. If* (1) *has a solution $y(t)$ which remains in a compact subset S_0 of S throughout $t_0 \leqslant t < \infty$ then* (1) *has a σ-periodic solution $u(t)$ such that $y(t) - u(t) \to 0$ as $t \to +\infty$.*

To illustrate the relation of Theorem 2 to Theorem 1 and to Massera's convergence theorem we apply it to the pair of uncoupled scalar equations

$$\frac{d\xi}{dt} = \phi(t, \xi), \qquad \frac{d\eta}{dt} = -\mu\eta, \tag{9}$$

in which the constant $\mu > 0$. This is a system of the form (1) with $x = \mathrm{col}(\xi, \eta)$ and $f(t, x) = \mathrm{col}(\phi(t, \xi), -\mu\eta)$. It is easy to verify that it satisfies (4) with $P = \mathrm{diag}(-1, 1)$, $\lambda = \mu - \varepsilon$, $S = \mathbb{R}^2$ provided that

$$\begin{aligned} &-(\mu - 2\varepsilon)(\xi_1 - \xi_2)^2 \\ &\quad \leqslant (\xi_1 - \xi_2)[\phi(t, \xi_1) - \phi(t, \xi_2)] \qquad \text{for } \xi_1, \xi_2 \in \mathbb{R}. \end{aligned} \tag{10}$$

In the special case when the partial derivative $\phi_\xi(t, \xi)$ exists and satisfies $-\mu < \inf \phi_\xi(t, \xi)$ in $\mathbb{R}^2$, the condition (10) holds for some sufficiently small constant $\varepsilon > 0$. Then (9) satisfies (H3), H(4) with $\lambda > 0$, $j = 1$, $S = \mathbb{R}^2$. Theorem 2, therefore, leads to a version of Massera's theorem for the scalar equation $d\xi/dt = \phi(t, \xi)$, with the extra restriction $-\mu < \inf \phi_\xi(t, \xi)$. This is satisfied in the special case when $\phi(t, \xi) \equiv \frac{1}{2}\mu \sin \xi$, for which the periodic solutions of (9) are the constant solutions $\xi = v\pi$, $\eta = 0$, with integer v. Hence, S may contain many different periodic solutions when the conditions of Theorem 2 hold. In this respect, Theorem 2 is more relaxed than Theorem 1 for which there can be only one periodic solution in S. When Theorem 1 is applied to (9) it leads similarly to a version of Massera's theorem for the scalar equation $d\xi/dt = \phi(t, \xi)$, with the extra restriction that either $0 > \sup \phi_\xi(t, \xi)$ or $0 < \inf \phi_\xi(t, \xi)$.

Proof of Theorem 2. In this proof Πx is a real-valued function because $j = 1$. Since the solutions $y(t), y(t + \sigma) \in S$ for $t \geqslant t_0$, it follows from (5) that $e^{2\lambda t} V(y(t) - y(t + \sigma))$ is monotonic decreasing in $[t_0, \infty)$. First, let us consider the case when $V(y(t_1) - y(t_1 + \sigma)) < 0$ for some $t_1 \geqslant t_0$. Then $V(y(t) - y(t + \sigma)) < 0$ for all $t \geqslant t_1$. This and (8) give

$$\begin{aligned} 2|\Pi(y(t) - y(t + \sigma))|^2 &> |M^{-1}(y(t) - y(t + \sigma))|^2 \\ &\geqslant |M|^{-2} |y(t) - y(t + \sigma)|^2, \end{aligned} \tag{11}$$

for all $t \geqslant t_1$. This shows that the scalar function $\Pi(y(t) - y(t + \sigma))$ is of constant sign in $[t_1, \infty)$. Then $\Pi y(t_1 + v\sigma) - \Pi y(t_1 + \sigma + v\sigma)$ has the same sign for all integers $v \geqslant 1$ and therefore $\{\Pi y(t_1 + v\sigma)\}$ is a monotonic sequence. It is also a bounded sequence because S_0 is compact. The series $\sum_{v=1}^{\infty} |\Pi(y(t_1 + v\sigma) - y(t_1 + \sigma + v\sigma))|$ therefore converges. Then (11) shows that $\sum_{v=1}^{\infty} |y(t_1 + v\sigma) - y(t_1 + \sigma + v\sigma)|$ converges. This implies that $\{y(t_1 + v\sigma)\}$ is a Cauchy sequence in $\mathbb{R}^n$. Hence, $y(t_1 + v\sigma) \to c$ as $v \to \infty$, where $c \in S_0$.

If the solution $u(t)$ of (1) having $u(t_1) = c$ lies in S throughout $t_1 \leqslant t < \alpha$ then $y(t + v\sigma) \to u(t)$ pointwise in $[t_1, \alpha)$ as $v \to \infty$. This solution $u(t)$ cannot leave the compact set S_0 in $[t_1, \alpha)$ because, if it did so, then some neighbouring solution $y(t + v\sigma)$ would also leave S_0, contradicting $y(t) \in S_0$ for $t \geqslant t_0$. Hence $u(t) \in S_0$ throughout $[t_1, \infty)$ and

$$u(t_1 + \sigma) = \lim_{v \to \infty} y(t_1 + \sigma + v\sigma) = \lim_{v \to \infty} y(t_1 + v\sigma) = c = u(t_1).$$

This shows that $u(t)$ is σ-periodic. Also $y(t)-u(t)\to 0$ as $t\to +\infty$ because $y(t_1+\nu\sigma)-u(t_1+\nu\sigma)=y(t_1+\nu\sigma)-u(t_1)\to 0$ as $\nu\to\infty$. The conclusion of Theorem 2 therefore holds provided that $V(y(t_1)-y(t_1+\sigma))<0$ for some $t_1\geqslant t_0$.

Now consider the case when $V(y(t)-y(t+\sigma))\geqslant 0$ for all $t\geqslant t_0$. Replacing $x(t)$ by $y(t+\sigma)$ in (6), we get

$$e^{2\lambda\theta}V(y(\theta)-y(\theta+\sigma))\geqslant 2\varepsilon\int_\theta^\tau e^{2\lambda t}|y(t)-y(t+\sigma)|^2\,dt,$$

for $\tau\geqslant\theta\geqslant t_0$. This and the Cauchy–Schwarz inequality give

$$\begin{aligned}\left[\int_\theta^\tau |y(t)-y(t+\sigma)|\,dt\right]^2 &\leqslant \int_\theta^\tau e^{2\lambda t}|y(t)-y(t+\sigma)|^2\,dt\int_\theta^\tau e^{-2\lambda t}dt,\\ &\leqslant (4\lambda\varepsilon)^{-1}V(y(\theta)-y(\theta+\sigma)).\end{aligned}$$

Hence, $\int_\theta^\infty |y(t)-y(t+\sigma)|\,dt$ converges for $\theta\geqslant t_0$.

In the left-hand side of (3) we replace $x(t)$, τ, θ by $y(t+\sigma)$, t, $\theta+\nu\sigma$, respectively, and then integrate with respect to t to get

$$|y(\theta+\nu\sigma)-y(\theta+\sigma+\nu\sigma)|K\leqslant\int_{\theta+\nu\sigma}^{\theta+\sigma+\nu\sigma}|y(t+\sigma)-y(t)|\,dt, \tag{12}$$

where the constant K is given by

$$K=\int_{\theta+\nu\sigma}^{\theta+\sigma+\nu\sigma}\exp[-\gamma(S_0)(t-\theta-\nu\sigma)]\,dt=\gamma(S_0)^{-1}(1-\exp[-\gamma(S_0)\,\sigma]).$$

Since $\int_\theta^\infty |y(t)-y(t+\sigma)|\,dt$ converges, (12) shows that the series $\sum_{\nu=1}^\infty |y(\theta+\nu\sigma)-y(\theta+\sigma+\nu\sigma)|$ converges and therefore $\{y(\theta+\nu\sigma)\}$ is a Cauchy sequence in $\mathbb{R}^n$. From this it follows, as in the previous case that $y(t)$ converges to a σ-periodic solution $u(t)$ such that $u(t)\in S_0$ for all t. This establishes Theorem 2.

The following corollary of Theorem 2 is needed later:

COROLLARY 2.1. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4) *with* $\lambda>0$, $j=1$. *If* (1) *has a solution* $z(t)$ *which remains in a compact subset* S_0 *of* S *throughout* $-\infty<t\leqslant t_0$ *then* (1) *has a* σ*-periodic solution* $w(t)$ *in* S_0 *such that* $z(t)-w(t)\to 0$ *as* $t\to -\infty$.

The corresponding corollary of Theorem 1 is also true but this has been omitted for the sake of brevity. Corollary 2.1 cannot be deduced from Theorem 2 by the familiar idea of replacing t by $-t$. This idea fails because (4) may become false when $f(t,x)$ is replaced by $-f(-t,x)$.

Proof of Corollary 2.1. Since S_0 is compact, $V(z(t)-z(t-\sigma))$ is bounded in $(-\infty, t_0]$ and $e^{2\lambda t}V(z(t)-z(t-\sigma))\to 0$ as $t\to -\infty$. It follows that $e^{2\lambda t}V(z(t)-z(t-\sigma))\leqslant 0$ in $(-\infty, t_0]$ because this function is monotonic decreasing by (5). It is strictly decreasing if $z(t)$ is not σ-periodic and then $V(z(t)-z(t-\sigma))<0$ in $(-\infty, t_0]$. This and (8) give

$$\begin{aligned} 2|\Pi(z(t)-z(t-\sigma))|^2 &> |M^{-1}(z(t)-z(t-\sigma))|^2 \\ &\geqslant |M|^{-2}|z(t)-z(t-\sigma)|^2, \end{aligned} \tag{13}$$

for all $t\leqslant t_0$. The scalar function $\Pi(z(t)-z(t-\sigma))$ is therefore of constant sign in $(-\infty, t_0]$. Hence $\{\Pi z(t_0-\nu\sigma)\}$ is a monotonic sequence. Since S_0 is compact, it is also a bounded sequence and the series

$$\sum_{\nu=1}^{\infty} |\Pi z(t_0-\nu\sigma)-\Pi z(t_0-\sigma-\nu\sigma)| \quad \text{converges.}$$

By (13),

$$\sum_{\nu=1}^{\infty} |z(t_0-\nu\sigma)-z(t_0-\sigma-\nu\sigma)| \quad \text{also converges.}$$

Then $\{z(t_0-\nu\sigma)\}$ is a Cauchy sequence in $\mathbb{R}^n$. From this it follows, as in the proof of Theorem 2, that (1) has a σ-periodic solution $w(t)$ such that $z(t)-w(t)\to 0$ as $t\to -\infty$. This establishes Corollary 2.1.

In general, it is difficult to give a qualitative description of the behaviour of all solution of (1) when $n>1$ because such equations may have recurrent solutions which are not periodic. Theorem 2 can help to simplify this problem as follows:

COROLLARY 2.2. *If* (1) *satisfies* (H1), (H2), (H3), (H4) *with* $\lambda>0$ *and* $j=1$ *then any recurrent solution* $y(t)$ *of* (1) *which lies wholly within a compact subset* S_0 *of* S *must be a* σ*-periodic solution. In particular,* (1) *has no proper subharmonic solutions in* S.

Proof. By Theorem 2, $y(t)$ converges to a σ-periodic solution $u(t)$. Since $y(t)$ is recurrent it is Poisson stable (see [14, p. 85]). That is, for each real τ there exists a strictly increasing sequence of positive integers $\{m(\nu)\}$ such that $y(\tau+m(\nu)\,\sigma)\to y(\tau)$ as $\nu\to\infty$. Since $u(\tau)=u(\tau+m(\nu)\,\sigma)$ this gives

$$y(\tau)-u(\tau)=\lim_{\nu\to\infty}\,[\,y(\tau+m(\nu)\,\sigma)-u(\tau+m(\nu)\,\sigma)]=0,$$

because $y(t)-u(t)\to 0$ as $t\to +\infty$. Hence $y(\tau)=u(\tau)$ for all real τ and therefore $y(t)$ is σ-periodic.

3. Stable Periodic Solution

The aim of this section is to add suitable hypotheses to Theorem 2 so as to ensure the existence of at least one stable periodic solution. We also obtain an analogue of a theorem of Pliss [10, p. 99] concerning the number of periodic solutions of an analytic equation. It is convenient to begin the discussion by assuming that (1) satisfies only (H1), (H3), (H4) with $\lambda > 0$ and $j \geqslant 1$.

A solution $x(t)$ of (1) is said to be *amenable* if $x(t) \in S$ throughout some interval $(-\infty, \theta]$ and $\int_{-\infty}^{\theta} e^{2\lambda t} |x(t)|^2 \, dt$ converges. Clearly any solution in S which is bounded in $(-\infty, \theta]$ is amenable. In particular any periodic solution which lies wholly in S is amenable.

Lemma 1. *If distinct amenable solutions $x_1(t)$, $x_2(t)$ lie in S throughout $(-\infty, t_0]$ then $V(x_1(t) - x_2(t)) < 0$ for all $t \leqslant t_0$. Conversely, if solutions $x(t)$, $y(t)$ lie in S and satisfy $V(x(t) - y(t)) \leqslant 0$ throughout $(-\infty, t_0]$ then $y(t)$ is amenable provided that $x(t)$ is amenable.*

This lemma is deduced from (H1), (H3) by the proof given in [16, p. 345]. It follows from (8) that the amenable solutions $x_1(t)$, $x_2(t)$ in Lemma 1 satisfy

$$2|\Pi x_1(t) - \Pi x_2(t)|^2 \geqslant |M^{-1}(x_1(t) - x_2(t))|^2 \geqslant |\Pi x_1(t) - \Pi x_2(t)|^2, \quad ((14)$$

for all $t \leqslant t_0$. This shows that if $\Pi x_1(t) = \Pi x_2(t)$ for one value of $t \leqslant t_0$ then $x_1(t) = x_2(t)$ for all $t \leqslant t_0$.

If $r \in \mathbb{R}$ let A_r denote the subset of S consisting of the points $x(r)$ taken over all those amenable solutions $x(t)$ of (1) which lie in S throughout $(-\infty, r]$. Then A_r is called an *amenable set* of (1) in S. Putting $t = r$ in (14) we get

$$|M^{-1}|^2 |p_1 - p_2|^2 \geqslant |\Pi p_1 - \Pi p_2|^2 \geqslant \tfrac{1}{2} |M|^{-2} |p_1 - p_2|^2, \qquad \text{for} \quad p_1, p_2 \in A_r. \tag{15}$$

Here $|M^{-1}|$, $|M|$ are any matrix norms which are consistent with the euclidean vector norm. This shows that mapping $\Pi: A_r \to \Pi A_r$ is one-to-one and bicontinuous. That is, A_r is homeomorphic to the subset ΠA_r of $\mathbb{R}^j$. So far, (H2) has not been assumed. When (H2) holds, $x(t - \sigma)$ is an amenable solution in S throughout $(-\infty, r + \sigma]$ if and only if $x(t)$ is an amenable solution in S throughout $(-\infty, r]$. This shows that $A_{r+\sigma} = A_r$, for all r. In the special case when (1) is autonomous, (H2) holds for all real σ. Then $A_r = A_0$ for all real r. For this special case A_0 is the same as the amenable set considered in [16, 17].

Now let us assume (H2) and also the following hypothesis:

(H5) *there exists a bounded open nonempty subset D of S with closure $\bar{D} \subset S$ such that if a solution $x(t)$ of* (1) *has $x(0) \in \bar{D}$ then $x(t) \in S$ for $0 \leqslant t \leqslant \sigma$ and $x(\sigma) \in D$.*

If $p \in S$ let $x(t, p)$ denote the solution of (1) having $x(0, p) = p$. Write

$$S_D = \{x(t, p) \in \mathbb{R}^n : p \in \bar{D} \text{ and } 0 \leqslant t \leqslant \sigma\}.$$

Then $S_D \subset S$, by (H5). Also S_D is compact, by (H1). If $p \in \bar{D}$ then (H5) ensures that $x(t, p)$ exists throughout $0 \leqslant t < \infty$ and satisfies $x(t, p) \in S_D$ for all $t \geqslant 0$. Let the period translation mapping $T: \bar{D} \to D$ be defined by $Tp = x(\sigma, p)$ for all p in $\bar{D}$. Then $Tp = p$, if and only if $x(t, p)$ is a σ-periodic solution.

Since $D \supset T\bar{D}$, we have $T^\nu \bar{D} \supset T^{\nu+1} \bar{D}$ for all integers $\nu \geqslant 0$. If B denotes the intersection of the decreasing sequence $\{T^\nu \bar{D}\}$ of compact sets then B is a non-empty compact subset of D such that $TB = B$. Since T is a one-to-one mapping there exists an inverse mapping $T^{-1}: B \to B$. This ensures that if $p \in B$ then $x(t, p)$ exists throughout $-\infty < t < \infty$ with $x(\nu\sigma, p) \in B$ for all integers ν. Then $x(t, p) \in S_D$ for all t and $x(t, p)$ is amenable because S_D is a compact subset of S. Hence, the amenable set $A_0 \supset B$ and the mapping $\Pi: B \to \Pi B$ is therefore homeomorphic.

The basic result of this section is as follows:

THEOREM 3. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4), (H5) *with $\lambda > 0$ and $j = 1$. Then* (1) *has at least one Lyapunov stable σ-periodic solution $x(t)$ such that $x(0) \in D$.*

For applications asymptotic stability is more significant than Lyapunov stability. However, the existence of an asymptotically stable periodic solution cannot be deduced from the hypotheses of Theorem 3. To see this we consider an autonomous system (9) in which $\phi(t, \xi) = \frac{1}{3}(\xi^{-1} - \xi)\mu$ when $\xi^2 > 1$ and $\phi(t, \xi) = 0$ when $-1 \leqslant \xi \leqslant 1$. In this case the periodic solutions of (9) are the constant solutions $\xi = k$, $\eta = 0$ with $-1 \leqslant k \leqslant 1$. These are all Lyapunov stable solutions which are not asymptotically stable. Since (9) satisfies the hypotheses of Theorem 3 with $S = \mathbb{R}^2$ and $D = \{(\xi, \eta) \in \mathbb{R}^2 : |\xi| < 2, \ |\eta| < 2\}$, we conclude that the existence of an asymptotically stable solution does not follows from these hypotheses. However, asymptotic stability can sometimes be deduced from Theorem 3 with the help of the following analogue of a result of Pliss [10, p. 97]:

THEOREM 4. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4) *with $\lambda > 0$ and $j = 1$. If* (1) *has an isolated periodic solution $x(t)$ in S which is Lyapunov stable then $x(t)$ is asymptotically stable.*

This result will be established first because its proof is more elementary than that of Theorem 3.

Proof of Theorem 4. It is sufficient to show that if a periodic solution $p(t)$ in S is Lyapunov stable but not asymptotically stable then $p(t)$ is not isolated. For each $\varepsilon > 0$ there eixsts $\delta(\varepsilon) > 0$ such that every solution $x(t)$ with $|x(0) - p(0)| < \delta(\varepsilon)$ satisfies $|x(t) - p(t)| < \varepsilon$, for $0 \leqslant t < \infty$. If ε is sufficiently small this solution $x(t)$ lies in a compact subset S_0 of S throughout $[0, \infty)$. By Theorem 2, $x(t)$ converges to a periodic solution $x_\omega(t)$ as $t \to +\infty$. Since $p(t)$ is not asymptotically stable we must have $x_\omega(t) \neq p(t)$ for all least one solution $x(t)$ with $|x(0) - p(0)| < \delta(\varepsilon)$. Since this satisfies $|x_\omega(t) - p(t)| \leqslant \varepsilon$ for all t, the periodic solution $p(t)$ is nonisolated. This establishes Theorem 4.

Proof of Theorem 3. First, we must identify a suitable periodic solution $g(t)$ of (1) and then prove that this $g(t)$ is a Lyapunov stable solution. Since $j = 1$, the mapping Πx is real-valued and $\mathbb{R} \supset \Pi A_0$. If $p \in A_0$, the amenable solution $x(t, p)$ exists throughout $-\infty < t \leqslant 0$ and we can define $T^{-1}p = x(-\sigma, p)$. Then $T^{-1}p \in A_0$ and this mapping $T^{-1}\colon A_0 \to A_0$ is an extension of the mapping $T^{-1}\colon B \to B$ defined above.

If $p_1, p_2 \in A_0$ and $p_1 \neq p_2$ the amenable solutions $x(t, p_1)$, $x(t, p_2)$ are distinct and satisfy (14) for all $t \leqslant 0$. This shows that $\Pi x(t, p_1) - \Pi x(t, p_2)$ does not vanish and is therefore of constant sign in $-\infty < t \leqslant 0$. If $\Pi p_1 \leqslant \Pi p_2$ it follows that $\Pi x(t, p_1) < \Pi x(t, p_2)$ for all $t \leqslant 0$. With $t = -\nu\sigma$ this gives

$$\Pi T^{-\nu} p_1 < \Pi T^{-\nu} p_2 \quad \text{if } p_1, p_2 \in A_0 \qquad \text{and} \qquad \Pi p_1 < \Pi p_2, \tag{16}$$

for all integers $\nu \geqslant 0$. This also holds for all integers $\nu < 0$ if $p_1, p_2 \in \bar{D} \cap A_0$.

We say that Πp is an *increasing point* of ΠA_0 if $p \in A_0$ and $\Pi p > \Pi T^{-1} p$. Then (16) gives $\Pi T^{-\nu} p > \Pi T^{-(\nu+1)} p$ for all integers $\nu \geqslant 0$ and therefore $\{\Pi T^{-\nu} p\}$ is a monotonic decreasing sequence which consists entirely of increasing points of ΠA_0. Similarly, Πp is a *decreasing point* of ΠA_0 if $p \in A_0$ and $\Pi p < \Pi T^{-1} p$. If $p, q \in A_0$ and $\Pi p \leqslant \Pi q < \Pi T^{-1} p$ then Πq is a decreasing point of ΠA_0 because (16) gives $\Pi q < \Pi T^{-1} p \leqslant \Pi T^{-1} q$. More generally,

$$\Pi q \text{ is a decreasing point of } \Pi A_0 \text{ if } \Pi p < \Pi q < \lim_{\nu \to +\infty} T^{-\nu} p, \tag{17}$$

because then $\Pi T^{-\nu} p \leqslant \Pi q < \Pi T^{-(\nu+1)} p$ for some integer $\nu \geqslant 0$.

Let B_1 denote the set of all p in B such that Πp is an increasing point of ΠA_0. Then $T^{-1} B_1 = B_1$ because $T^{-1} B = B$. Since $\Pi B_1 \subset \Pi B$, the compact set ΠB contains the number $\sup \Pi B_1$, provided that B_1 is not empty. Then there exists g_1 in B such that $\Pi g_1 = \sup \Pi B_1$. From (16) we get

$$\Pi T^{-1} g_1 = \sup \Pi T^{-1} B_1 = \sup \Pi B_1 = \Pi g_1.$$

Since $\Pi\colon B \to B$ is homeomorphic, this gives $T^{-1}g_1 = g_1$ and therefore $x(t, g_1)$ is a σ-periodic solution. Similarly, there exists g_2 in B such that $\Pi g_2 = \inf \Pi B$ and $T^{-1}g_2 = g_2$. Then $x(t, g_2)$ is also a σ-periodic solution. If B_1 is not empty we define $g(t) = x(t, g_1)$ and if B_1 is empty we define $g(t) = x(t, g_2)$. To complete the proof of Theorem 3 we must show that this σ-periodic solution $g(t)$ is Lyapunov stable. We do this by assuming that $g(t)$ is not Lyapunov stable and deducing a contradiction.

Before beginning that we prove the following result:

$$\textit{if any solution } z(t) \textit{ has } 0 = \lim_{t \to -\infty} [z(t) - g(t)] \textit{ then } z(t) \equiv g(t). \tag{18}$$

Since $g(-\nu\sigma) = g(0)$ for all integers ν we have

$$0 = \lim_{\nu \to +\infty} [g(-\nu\sigma) - z(-\nu\sigma)] = g(0) - \lim_{\nu \to +\infty} z(-\nu\sigma). \tag{19}$$

Since D is open and $g(0) \in D$ this shows that $z(-\nu\sigma) \in D$ for all large integers $\nu > 0$. Then (H5) ensures that $z(t)$ exists throughout $-\nu\sigma \leqslant t < \infty$ and has $z(0) \in T^\nu \bar{D}$. Since this holds for all large $\nu > 0$ we have $z(0) \in B$ and therefore $z(t)$ is amenable. If we assume that $\Pi z(0) > \Pi g(0)$ then $\Pi z(0)$ is an increasing point of ΠA_0 because $\Pi g(0) = \lim \Pi z(-\nu\sigma)$, by (19). Then $z(0) \in B_1$ and therefore $\Pi z(0) \leqslant \sup \Pi B_1 = \Pi g_1 = \Pi g(0) < \Pi z(0)$. This contradiction proves that $\Pi z(0) \not> \Pi g(0)$. If we assume that $\Pi z(0) < \Pi g(0)$ then $\Pi g(0) \neq \inf \Pi B$ because $z(0) \in B$. Then $\Pi g(0) = \sup \Pi B_1$ and there exists q in B_1 such that $\Pi g(0) > \Pi q > \Pi z(0)$. This and (17) show that Πq is a decreasing point of ΠA_0 because $g(0) = \lim z(-\nu\sigma) = \lim T^{-\nu} z(0)$. Since this contradicts $q \in B_1$ we deduce that $\Pi z(0) \not< \Pi g(0)$. Hence $\Pi z(0) = \Pi g(0)$. Since $\Pi\colon A_0 \to A_0$ is homeomorphic, we get $z(0) = g(0)$ and therefore $z(t) \equiv g(t)$. This establishes (18).

Now suppose that the solution $g(t)$ is not Lyapunov stable. Then for all sufficiently small $\varepsilon > 0$ and all $\delta > 0$ there exists a solution $x(t)$ such that $|x(0) - g(0)| < \delta$, $|x(\tau) - g(\tau)| = \varepsilon$ and $|x(t) - g(t)| < \varepsilon$ for $0 \leqslant t < \tau$. Since $g(0) \in D$ and D is open there exists $\delta_0 > 0$ such that $x(0) \in D$ when $0 < \delta < \delta_0$. Then (H5) ensures that $x(t) \in S_D$ for $0 \leqslant t \leqslant \tau$. Since S_D is a compact subset of S, we deduce from (3) that

$$\varepsilon = |x(\tau) - g(\tau)| \leqslant |x(m\sigma) - g(m\sigma)| \exp[\sigma\gamma(S_D)],$$

where m is the integer satisfying $m\sigma \leqslant \tau < \sigma + m\sigma$. Write $y(t, \varepsilon, \delta) = x(t + m\sigma)$. Then $y(t, \varepsilon, \delta)$ is a solution of (1) in $-m\sigma \leqslant t \leqslant 0$ such that

$$y(t, \varepsilon, \delta) \in S_D \quad \text{and} \quad |y(t, \varepsilon, \delta) - g(t)| \leqslant \varepsilon \quad \text{for } -m\sigma \leqslant t \leqslant 0, \tag{20}$$

$$|y(-m\sigma, \varepsilon, \delta) - g(0)| < \delta, \qquad \varepsilon \geqslant |y(0, \varepsilon, \delta) - g(0)| \geqslant \varepsilon \exp[-\sigma\gamma(S_D)]. \tag{21}$$

Here the integer m depends on ε, δ, and $m(\varepsilon, \delta) \to +\infty$ as $\delta \to 0+$ keeping ε fixed.

Since $y(0, \varepsilon, \delta) = x(m\sigma) \in D$, Weierstrass's theorem enables us to choose $c(\varepsilon)$ in $\bar{D}$ and a sequence $\{\delta_\nu\}$ such that $y(0, \varepsilon, \delta_\nu) \to c(\varepsilon)$ and $\delta_\nu \to 0$ as $\nu \to \infty$. Let $z_\varepsilon(t)$ denote the solution of (1) having $z_\varepsilon(0) = c(\varepsilon)$. Then $y(t, \varepsilon, \sigma_\nu) \to z_\varepsilon(t)$ pointwise in $-\infty < t \leqslant 0$ as $\nu \to \infty$ because solutions in S vary continuously with initial value. From (20), (21) we deduce that

$$z_\varepsilon(t) \in S_D \qquad \text{and} \qquad |z_\varepsilon(t) - g(t)| \leqslant \varepsilon \quad \text{for } -\infty < t \leqslant 0, \tag{22}$$

$$g(0) = \lim_{\nu \to \infty} y(-m(\varepsilon, \delta_\nu)\,\sigma, \varepsilon, \delta_\nu), \qquad \varepsilon \geqslant |z_\varepsilon(0) - g(0)| \geqslant \varepsilon \exp[-\sigma\gamma(S_D)]. \tag{23}$$

Since compact $S_D \subset S$, (22) shows that $z_\varepsilon(t)$ is an amenable solution. By Corollary 2.1 there exists a σ-periodic solution $w_\varepsilon(t)$ in S_D such that $w_\varepsilon(t) - z_\varepsilon(t) \to 0$ as $t \to -\infty$. Then $w_\varepsilon(\tau) = w_\varepsilon(\tau - \nu\sigma)$ for all integers ν and

$$0 = \lim_{\nu \to +\infty} [w_\varepsilon(\tau - \nu\sigma) - z_\varepsilon(\tau - \nu\sigma)] = w_\varepsilon(\tau) - \lim_{\nu \to +\infty} z_\varepsilon(\tau - \nu\sigma). \tag{24}$$

If we assume that $w_\varepsilon(t) \equiv g(t)$ then (18) gives $z_\varepsilon(t) \equiv g(t)$ which contradicts (23). Hence $w_\varepsilon(t) \neq g(t)$ for all t and Lemma 1 gives

$$0 > V(w_\varepsilon(t) - g(t)) \qquad \text{for } -\infty < t < \infty. \tag{25}$$

If we substitute $t = \tau - \nu\sigma$ in (22) and make $\nu \to \infty$ then (24) gives

$$0 < |w_\varepsilon(\tau) - g(\tau)| \leqslant \varepsilon \qquad \text{for } -\infty < \tau < \infty. \tag{26}$$

Since $z_\varepsilon(t)$ and $g(t)$ are distinct amenable solutions, it follows from (14) the function $\Pi z_\varepsilon(t) - \Pi g(t)$ does not vanish and is therefore of constant sign in the interval $(-\infty, 0]$. This and (24) give

$$\begin{aligned} &\operatorname{sign} \Pi[z_\varepsilon(0) - g(0)] \\ &\qquad = \operatorname{sign} \Pi[z_\varepsilon(-\nu\delta) - g(-\nu\delta)] = \operatorname{sign} \Pi[w_\varepsilon(0) - g(0)]. \end{aligned}$$

The real numbers $\Pi z_\varepsilon(0)$, $\Pi w_\varepsilon(0)$ therefore lie on the same side of $\Pi g(0)$.

Since a periodic solution $w_\varepsilon(t)$ exists for each small $\varepsilon > 0$, we deduce from (26) that the periodic solution $g(t)$ is nonisolated and $\Pi w_\varepsilon(0) \to \Pi g(0)$ as $\varepsilon \to 0+$. We can therefore choose small positive ε, η such that $\Pi w_\eta(0)$ lies on the same side of $\Pi g(0)$ as $\Pi w_\varepsilon(0)$ and strictly between $\Pi g(0)$ and $\Pi z_\varepsilon(0)$. Then

$$0 > [\Pi w_\eta(0) - \Pi g(0)][\Pi w_\eta(0) - \Pi z_\varepsilon(0)].$$

Since (23) gives $g(0) = \lim y(-m\sigma, \varepsilon, \delta_\nu)$, this inequality can be written as

$$0 > \lim_{\nu \to \infty} [\Pi w_\eta(-m\sigma) - \Pi y(-m\sigma, \varepsilon, \delta_\nu)][\Pi w_\eta(0) - \Pi y(0, \varepsilon, \sigma_\nu)], \quad (27)$$

because $z_\varepsilon(0) = c(\varepsilon) = \lim y(0, \varepsilon, \delta_\nu)$, by definition of $z_\varepsilon(t)$. Also, (25) gives

$$0 > V(w_\eta(0) - g(0)) = \lim_{\nu \to \infty} V(w_\eta(-m\sigma) - y(-m\sigma, \varepsilon, \delta_\nu)). \quad (28)$$

It follows from (27) that the function $\Pi w_\eta(t) - \Pi y(t, \varepsilon, \delta_\nu)$ is of opposite sign at the endpoints of the interval $-m\sigma \leqslant t \leqslant 0$, provided that ν is sufficiently large. Hence,

$$\Pi w_\eta(t_\nu) = \Pi y(t_\nu, \varepsilon, \delta_\nu) \qquad \text{for some } t_\nu \text{ with } -m(\varepsilon, \delta_\nu)\,\sigma < \tau_\nu < 0. \quad (29)$$

The function $e^{2\lambda t} V(w_\eta(t) - y(t, \varepsilon, \delta_\nu))$ is monotonic decreasing in the interval $-m\sigma \leqslant t \leqslant 0$, by (5). It is negative throughout this interval because (28) shows that it is negative when $t = -m\sigma$, provided that ν is large. If we substitute $x = x_\eta(t) - y(t, \varepsilon, \delta_\nu)$ in (8), this gives

$$2|\Pi w_\eta(t) - \Pi y(t, \varepsilon, \delta_\nu)|^2 \geqslant |M^{-1}(w_\eta(t) - y(t, \varepsilon, \delta_\nu))|^2,$$

for $-m\sigma \leqslant t \leqslant 0$. This and (29) give $w_\eta(t_\nu) - y(t_\nu, \varepsilon, \delta_\nu) = 0$, which shows that the solutions $w_\eta(t)$ and $y(t, \varepsilon, \delta_\nu)$ coincide throughout $-m\sigma \leqslant t \leqslant 0$. This and (23) give

$$g(0) = \lim_{\nu \to \infty} y(-m\sigma, \varepsilon, \delta_\nu) = \lim_{\nu \to \infty} w_\eta(-m\sigma) = w_\eta(0), \quad (30)$$

because $w_\eta(-m\sigma) = w_\eta(0)$ for all ν. Since (30) contradicts (26) we conclude that the solution $g(t)$ must be Lyapunov stable. This finishes the proof of Theorem 3.

Now, suppose that $f(t, x)$ is differentiable with respect to x in $\mathbb{R} \times S$. Then the $n \times n$ Jacobian matrix $J(t, x) = \partial f / \partial x$ exists in $\mathbb{R} \times S$ and satisfies

$$J(t, x)\, v = \lim_{h \to 0} h^{-1}[f(t, x + hv) - f(t, x)], \quad (31)$$

for each v in $\mathbb{R}^n$. If $x \in S$ then $x + hv \in S$ for all sufficiently small $h \neq 0$. Substitute $x_1 = x + hv$, $x_2 = x$ in (4) and then divide it by h^2 to get

$$v^* P[\lambda v + h^{-1}[f(t, x + hv) - f(t, x)]] \leqslant -\varepsilon |v|^2.$$

This and (31) give $v^*P[\lambda v + J(t, x) v] \leqslant -\varepsilon|v|^2$ for all v in $\mathbb{R}^n$. From the symmetric matrix of this quadratic form we get

$$PJ(t, x) + J(t, x)^* P + 2\lambda P < 0, \tag{32}$$

where the inequality means that the matrix is negative definite.

The function $f(t, x)$ is said to be analytic at (t_0, x_0) if there exists a neighbourhood of (t_0, x_0) throughout which it is the sum of a convergent multiple power series in the variables $t - t_0$, $x - x_0$. If $f(t, x)$ is analytic at every point of $\mathbb{R} \times S$ then (H1) and (32) hold. Also $Tp = x(\sigma, p)$ is an analytic function of p in $\bar{D}$. The following theorem is analogous to a result of Pliss [10, p. 99]:

THEOREM 5. *Suppose that* (1) *satisfies* (H2), (H3), (H4), (H5) *with* $\lambda > 0$ *and* $j = 1$. *If* $f(t, x)$ *is analytic in* $\mathbb{R} \times S$ *then* (1) *has only a finite number of periodic solutions* $x(t)$ *such that* $x(0) \in \bar{D}$. *Furthermore, at least one of these periodic solutions is asymptotically stable.*

Proof. By (H5), $x(t, p) \in S$ if $p \in \bar{D}$ and $0 \leqslant t \leqslant \sigma$. The $n \times n$ matrix $H(t, p) = \partial x(t, p)/\partial p$ therefore exists in $[0, \sigma] \times \bar{D}$. It has $H(0, p) = I$ because $x(0, p) = p$. Also $H(\sigma, p)$ is the Jacobian matrix $\partial Tp/\partial p$ because $Tp = x(\sigma, p)$. Substitute $x(t, p)$ in (1) and then differentiate it with respect to p to get

$$\partial H(t, p)/\partial t = J(t, x(t, p))\, H(t, p). \tag{33}$$

By Theorem 3 there exists at least one σ-periodic solution $x(t)$ such that $x(0) \in \bar{D}$. Any such solution is of the form $x(t, c)$ with $c \in \bar{D}$ and $Tc = c$. Then $c \in B$ because $c = T^\nu c \in T^\nu \bar{D}$ for all integers $\nu \geqslant 0$. If we put $p = c$ in (33) then $J(t, x(t, c))$ is a σ-periodic matrix which satisfies (32). Under these conditions, [17, Theorem 1] asserts that the matrix $H(\sigma, c)$ has exactly j eigenvalues β with $|\beta| > e^{-\lambda\sigma}$, where j is the number of negative eigenvalues of P. Since $j = 1$ in Theorem 5, the real matrix $H(\sigma, c)$ has only one such eigenvalue β and this is real. Then the real version of Jordan's theorem (see [4, p. 81]) gives

$$N^{-1} H(\sigma, c)\, N = \operatorname{diag}(\beta, b), \tag{34}$$

where N is an invertible real $n \times n$ matrix and b is a real $(n-1) \times (n-1)$ block whose eigenvalues z all have $|z| \leqslant e^{-\lambda\sigma}$.

Now, let us assume that, contrary to Theorem 5, B contains an infinite set of fixed points c of T. If E denotes the subset of B consisting of all nonisolated fixed points of T then E is a nonempty closed set and ΠE is a compact subset of $\mathbb{R}$. Since $\mathbb{R}$ is connected, ΠE cannot be an open set. We now obtain a contradiction by proving that ΠE is open.

Since $\partial Tp/\partial p = H(\sigma, p)$, the Taylor expansion of the analytic function $p - Tp$ about any point c of E gives

$$p - Tp = [I - H(\sigma, c)](p - c) + \Psi(p),$$

where $\Psi(p)$ is analytic in $\bar{D}$ and $|\Psi(p)| = O(|p-c|^2)$ as $p \to c$. This and (34) show that the equation $p = Tp$ is equivalent to

$$N^{-1}\Psi(p) = N^{-1}[H(\sigma, c) - I](p - c) = \operatorname{diag}(\beta - 1, b - I)\, N^{-1}(p - c).$$

Substitute $p = c + N \operatorname{col}(\xi, \eta)$ to reduce this to the pair of equations

$$\psi_1(\xi, \eta) = (\beta - 1)\,\xi, \qquad \psi_2(\xi, \eta) = (b - I)\,\eta, \tag{35}$$

where $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^{n-1}$ and the analytic functions ψ_1, ψ_2 are $O(\xi^2 + \eta^2)$ as $(\xi, \eta) \to (0, 0)$. Since $\det(b - I) \neq 0$, the analytic version of the implicit function theorem shows that the second equation of (35) has a solution $\eta(\xi)$ which is analytic in some interval $-\delta < \xi < \delta$ with $\eta(0) = 0$. Furthermore this is the only solution which is near $\eta = 0$ when $-\delta < \xi < \delta$. All the solutions of (35) near (0, 0) and therefore of the form $(\xi, \eta(\xi))$, where ξ is a zero in $(-\delta, \delta)$ of the function $(\beta - 1)\,\xi - \psi_1(\xi, \eta(\xi))$. This function has a nonisolated zero at $\xi = 0$ because c is a nonisolated fixed point of T. Since it is analytic at $\xi = 0$, this function must be identically zero in some open interval $-\delta_0 < \xi < \delta_0$ and therefore $(\xi, \eta(\xi))$ defines a continuum of solutions of (35) through (0, 0). Hence, c belongs to a continuum of fixed points of T of the form $p_\xi = c + N \operatorname{col}(\xi, \eta(\xi))$. Clearly $p_\xi \in E$ for $-\delta_0 < \xi < \delta_0$ and the points Πp_ξ cover an open interval of $\mathbb{R}$ which contains Πc. That is, Πc is an interior point of ΠE for each c in E. The compact set ΠE is therefore an open set.

Since this contradicts the connectedness of $\mathbb{R}$ we deduce that T has only a finite number of fixed points in B. Hence (1) has only a finite number of periodic solutions $x(t)$ with $x(0) \in D$ and each of these is isolated. In particular the Lyapunov stable periodic solution given by Theorem 3 is isolated and therefore asymptotically stable by Theorem 4. This establishes Theorem 5.

Suppose now that (H1) holds with $S = \mathbb{R}^n$. Then (1) is said to be *dissipative* if there exist a constant k and a positive function $\tau(\rho)$ defined for all $\rho > 0$ such that every solution $x(t)$ with $|x(t_0)| \leqslant \rho$ exists throughout $t_0 \leqslant t < \infty$ and satisfies $|x(t)| < k$ for all $t \geqslant t_0 + \tau(\rho)$. The number k is called the *ultimate bound* of (1). Sufficient conditions for (1) to be dissiative are discussed in [11, p. 45]. In particular, Pliss [10, p. 42] showed that (1) is dissipative if there exists a constant $n \times n$ matrix L such that

$$0 = \lim_{x \to \infty} |x|^{-1}[f(t, x) - Lx] \qquad \text{uniformly in } -\infty < t < \infty, \tag{36}$$

and re $z<0$ for all eigenvalues z of L. The following theorem is related to a result of Yoshizawa [18]:

THEOREM 6. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4) *with* $S=\mathbb{R}^n$, $\lambda>0$, $j=1$. *If* (1) *is dissipative then each solution* $x(t)$ *converges to a* σ-*periodic solution* $x_\omega(t)$ *as* $t\to+\infty$ *and at least one periodic solution is Lyapunov stable. If, in addition,* $f(t,x)$ *is analytic in* $\mathbb{R}\times\mathbb{R}^n$ *then* (1) *has only a finite number of periodic solutions and at least one of these is asymptotically stable.*

Yoshizawa [18] proved the existence of at least one σ-periodic solution without using (H3), (H4) but his result provides no information about convergence, stability or number of periodic solutions.

Proof of Theorem 6. Let $D=\{x\in\mathbb{R}^n\colon |x|<1+k\}$, where k is the ultimate bound. If a solution $x(t)$ has $x(0)\in\bar{D}$ then $x(m\sigma)\in D$, where m is the least integer exceeding $\sigma^{-1}\tau(1+k)$. That is, (1) satisfies (H5) except that σ is replaced by the period $m\sigma$. Theorem 3 then shows that (1) has at least one Lyapunov stable $m\sigma$-periodic solution. However every $m\sigma$-periodic solution is σ-periodic by Corollary 2.2. The other assertions of Theorem 6 follow similarly from Theorems 2 and 5.

4. MASSERA'S SECOND THEOREM

In this section (H1) holds with $S=\mathbb{R}^n$. The following is also assumed

(H6) *each solution of* (1) *has interval of existence of the form* (θ,∞).

Theorem 2 of Massera [9] states that if a differential equation in $\mathbb{R}^2$ satisfies (H1), (H2), (H6) and has a solution $y(t)$ which is bounded in some interval $[t_0,\infty)$ then it has at least one σ-periodic solution $u(t)$. This theorem provides no explicit relation between $y(t)$ and $u(t)$; in general $y(t)$ does not converge to $u(t)$ as $t\to+\infty$. Massera also showed that this theorem cannot be extended to higher-dimensional differential equations without adding further hypotheses to it. For equations in $\mathbb{R}^n$, Halanay (see [11, p. 74]) proved a similar result which omits (H6) but requires the bounded solution $y(t)$ to satisfy the extra condition $y(\sigma+\nu\sigma)-y(\nu\sigma)\to 0$ as the integer $\nu\to+\infty$. However, this extra condition is not always easy to verify in practice. In the present section our main aim is to prove the following result:

THEOREM 7. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4), (H6) *with* $S=\mathbb{R}^n$, $\lambda>0$, $j=2$. *If* (1) *has a solution* $y(t)$ *which is bounded in some interval* $[t_0,\infty)$ *then* (1) *has at least one* σ-*periodic solution* $u(t)$.

This can be regarded as an analogue of Massera's second theorem in which the condition $n=2$ has been replaced by $j=2$. Our proof of Theorem 7 is based on the following result which concerns the amenable sets A_r defined in Section 3:

THEOREM 8. *Suppose that* (1) *satisfies* (H1), (H3), (H4), (H6) *with* $S=\mathbb{R}^n$, $\lambda>0$, $j\geqslant 1$. *If* (1) *has at least one amenable solution then* $\Pi A_r=\mathbb{R}^j$ *for all real r and the restricted mapping* $\Pi\colon A_r\to\mathbb{R}^j$ *is homeomorphic.*

This result generalises [16, Theorem 3] which concerns the special case when (1) is autonomous. Indeed, it improves [16, Theorem 3] because that result replaces (H6) by the more restrictive assumption that all solutions of (1) exist throughout $-\infty<t<\infty$. Our proof of Theorem 8 is based on the following lemma:

LEMMA 2. *Suppose that* $y(t)$ *is an amenable solution of* (1). *If* $\zeta\in\mathbb{R}^j$ *and* θ, r *are any real numbers with* $\theta<r$ *then there exists a solution* $z_\theta(t)$ *in* $\theta\leqslant t<\infty$ *such that* $\zeta=\Pi z_\theta(r)$ *and* $V(z_\theta(t)-y(t))\leqslant 0$ *throughout* $[\theta,\infty)$.

Proof of Lemma 2. If $X\in\mathbb{R}^j$ let $x(t,X,\theta)$ denote that solution $x(t)$ of (1) which has $x(\theta)=y(\theta)+M\operatorname{col}(X,0)$. Here M is the matrix in (8) and $y(\theta)$ is the value at $t=\theta$ of the given amenable solution $y(t)$. By (H6), the solution $x(t,X,\theta)$ exists for $\theta\leqslant t<\infty$. When $X=0$ it reduces to $x(t,0,\theta)=y(t)$. Also

$$x(\theta,X_1,\theta)-x(\theta,X_2,\theta)=M\operatorname{col}(X_1-X_2,0).$$

The relation $V(M\operatorname{col}(X,Y))=Y^2-X^2$ was used in Section 2 to prove (8). For all X_1, X_2 in $\mathbb{R}^j$ this relation shows that

$$-|X_1-X_2|^2=V(M\operatorname{col}(X_1-X_2,0))=V(x(\theta,X_1,\theta)-x(\theta,X_2,\theta)).$$

By (5), $e^{2\lambda t}V(x(t,X_1,\theta)-x(t,X_2,\theta))$ is decreasing and therefore

$$-e^{2\lambda\theta}|X_1-X_2|^2\geqslant e^{2\lambda t}V(x(t,X_1,\theta)-x(t,X_2,\theta))) \qquad \text{for all } t\geqslant\theta. \tag{37}$$

Hence $V(x(t,X_1,\theta)-x(t,X_2,\theta))\leqslant 0$ and (8) gives

$$2|\Pi(x(t,X_1,\theta)-x(t,X_2,\theta))|^2\geqslant|M^{-1}(x(t,X_1,\theta)-x(t,X_2,\theta))|^2, \tag{38}$$

for $t\geqslant\theta$. From (8), $|\Pi x|^2\geqslant -V(x)$. This and (37) give

$$e^{2\lambda t}|\Pi(x(t,X_1,\theta)-x(t,X_2,\theta))|^2\geqslant e^{2\lambda\theta}|X_1-X_2|^2 \qquad \text{for } t\geqslant\theta. \tag{39}$$

For each $\theta<r$ let $g_\theta\colon \mathbb{R}^j\to\mathbb{R}^j$ be the continuous mapping defined by $g_\theta(X)=\Pi x(r, X, \theta)$ for all X in $\mathbb{R}^j$. With $t=r$, (39) gives

$$e^{2\lambda r}|g_\theta(X_1)-g_\theta(X_2)|^2\geqslant e^{2\lambda\theta}|X_1-X_2|^2 \qquad \text{for } X_1,X_2\in\mathbb{R}^j. \tag{40}$$

This shows that g_θ is one-to-one and therefore $g_\theta(\mathbb{R}^j)$ is an open subset of $\mathbb{R}^j$ by Brouwer's theorem on invariance of domain (see [8, p. 50]). We now prove that $g_\theta(\mathbb{R}^j)=\mathbb{R}^j$ by the method of contradiction. If we suppose that $g_\theta(\mathbb{R}^j)$ is not the whole of $\mathbb{R}^j$ then it has a boundary point b in $\mathbb{R}^j$. Then $b=\lim g_\theta(X_\nu)$ for some sequence $\{X_\nu\}$ in $\mathbb{R}^j$. Since $\{g_\theta(X_\nu)\}$ is a Cauchy sequence (40) shows that $\{X_\nu\}$ is also a Cauchy sequence and therefore $X_\nu\to a$, where $a\in\mathbb{R}^j$. Then $b=\lim g_\theta(X_\nu)=g_\theta(a)$ because g_θ is continuous at a. Hence $b\in g_\theta(\mathbb{R}^j)$ and therefore b is an interior point of this open set. Since this contradicts the supposition that b is a boundary point of $g_\theta(\mathbb{R}^j)$ we conclude that $g_\theta(\mathbb{R}^j)=\mathbb{R}^j$.

This ensures that if $\zeta\in\mathbb{R}^j$ there exists a point $v(\theta)$ in $\mathbb{R}^j$ such that $\zeta=g_\theta(v(\theta))=\Pi x(r, v(\theta), \theta)$. If we write $z_\theta(t)=x(t, v(\theta), \theta)$ then $\zeta=\Pi z_\theta(r)$ and $z_\theta(t)$ satisfies (1) in $[\theta, \infty)$. Since $x(t, 0, \theta)=y(t)$ we can put $X_1=v(\theta)$, $X_2=0$ in (37) to get $V(z_\theta(t)-y(t))\leqslant 0$ for all $t\geqslant\theta$. This establishes Lemma 2.

Proof of Theorem 8. To prove $\Pi A_r=\mathbb{R}^j$ it is sufficient to show that for each ζ in $\mathbb{R}^j$ there exists an amenable solution $u(t)$ of (1) such that $\zeta=\Pi u(r)$. Such a solution $u(t)$ will be obtained as the limit of a suitable sequence of the solutions $z_\theta(t)=x(t, v(\theta), \theta)$ found in Lemma 2. Since $\zeta=\Pi z_\theta(r)=\Pi x(r, v(\theta), \theta)$ and $x(t, 0, \theta)=y(t)$, we can put $X_1=v(\theta)$, $X_2=0$, $t=r$ in (38) to get

$$\begin{aligned} 2|\zeta-\Pi y(r)|^2&\geqslant |M^{-1}(x(r, v(\theta), \theta)-y(r))|^2,\\ &\geqslant |M|^{-2}|x(r, v(\theta), \theta)-y(r)|^2 \quad \text{for } \theta\leqslant r. \end{aligned} \tag{41}$$

Putting $x(t)=x(t, v(\theta), \theta)$, $\tau=r$ in (6) we get

$$-e^{2\lambda r}V(x(r, v(\theta), \theta)-y(r))\geqslant 2\varepsilon\int_\theta^r e^{2\lambda t}|x(t, v(\theta), \theta)-y(t)|^2\,dt,$$

because $V(x(\theta, v(\theta), \theta)-y(\theta))\leqslant 0$, by (37). This and (41) give

$$\begin{aligned} \int_\theta^r e^{2\lambda t}|y(t)-x(t, v(\theta), \theta)|^2\,dt&\leqslant(2\varepsilon)^{-1}e^{2\lambda r}|x(r, v(\theta), \theta)-y(r)|^2|P|,\\ &\leqslant\varepsilon^{-1}e^{2\lambda r}|\zeta-\Pi y(r)|^2|M|^2|P|, \end{aligned} \tag{42}$$

for all $\theta\leqslant r$. By (41), $|x(r, v(\theta), \theta)|$ is bounded for $\theta\leqslant r$. A sequence $\{\theta_\nu\}$ can therefore be chosen such that $x(r, v(\theta_\nu), \theta_\nu)\to q$ and $\theta_\nu\to-\infty$ as

$\nu \to \infty$, where $q \in \mathbb{R}^n$. If $u(t)$ denotes the solution having $u(r) = q$ then $u(t)$ exists in $[r, \infty)$, by (H6). Also $\Pi u(r) = \zeta$ because $\zeta = \Pi x(r, v(\theta_\nu), \theta_\nu)$ for all ν.

We now prove that $u(t)$ also exist throughout $(-\infty, r]$. For this it is sufficient to prove that it exists throughout $[\beta, r]$ for every $\beta < r$. When ν is sufficiently large, $\theta_\nu \leqslant \beta - 1$ and (42) gives

$$e^{2\lambda(\beta-1)}|y(t_\nu) - x(t_\nu, v(\theta_\nu), \theta_\nu)|^2 \leqslant \varepsilon^{-1} e^{2\lambda r}|\zeta - \Pi y(r)|^2 |M|^2 |P|,$$

Applying the mean value theorem to this integral, we get

$$e^{2\lambda(\beta-1)}|y(t_\nu) - x(t_\nu, v(\theta_\nu), \theta_\nu)|^2 \leqslant \varepsilon^{-1} e^{2\lambda r}|\zeta - \Pi y(r)|^2 |M|^2 |P|,$$

for some number t_ν in $[\beta - 1, \beta]$. When β is fixed this shows that t_ν and $|x(t_\nu, v(\theta_\nu), \theta_\nu)|$ are both bounded for all large ν. By the Weierstrass subsequence theorem we can suppose that $t_\nu \to l$ and $x(t_\nu, v(\theta_\nu), \theta_\nu) \to p$ as $\nu \to \infty$, where $l \in [\beta - 1, \beta]$ and $p \in \mathbb{R}^n$. If $w(t)$ denotes the solution of (1) having $w(l) = p$ then $w(t)$ exists in $[l, \infty)$ by (H6). Since solutions vary continuously with their initial values, $x(t, v(\theta_\nu), \theta_\nu) \to w(t)$ pointwise in $[l, \infty)$ as $\nu \to \infty$. In particular,

$$w(r) = \lim x(r, v(\theta_\nu), \theta_\nu) = q = u(r)$$

and therefore $w(t)$ is an extension of $u(t)$ throughout $[l, r]$. Since $l \leqslant \beta$, $u(t)$ exists in $[\beta, \infty)$ for each $\beta < r$. Hence $u(t)$ exists throughout $(-\infty, \infty)$.

It only remains to prove that $u(t)$ is amenable. For $t \geqslant \theta_\nu$, (37) gives $0 \geqslant V(y(t) - x(t, v(\theta_\nu), \theta_\nu))$. When $\nu \to \infty$, this gives $0 \geqslant V(y(t) - w(t))$ for $t \geqslant l$. Since $w(t)$ coincides with $u(t)$ this gives $0 \geqslant V(y(t) - u(t))$ in $[\beta, \infty)$ for each $\beta < r$. That is $0 \geqslant V(y(t) - u(t)$ throughout $(-\infty, \infty)$. Since $y(t)$ is amenable, Lemma 1 shows that $u(t)$ is also amenable. Then $u(r) \in A_r$. We proved above that $\zeta = \Pi u(r)$ and therefore $\zeta \in \Pi A_r$ for each ζ in $\mathbb{R}^j$. That is, $\mathbb{R}^j = \Pi A_r$. Since we proved in Section 3 that the mapping Π: $A_r \to \Pi A_r$ is homeomorphic, this completes the proof of Theorem 8.

Our applications of Theorem 8 are based on the following corollary:

COROLLARY 8.1. *Suppose that* (1) *satisfies* (H1), (H3), (H4), (H6) *with* $S = \mathbb{R}^n$, $\lambda > 0$ *and* $j \geqslant 1$. *Suppose also that* (1) *has at least one amenable solution. Then there exists a continuous function* $\phi(t, \zeta)$ *from* $\mathbb{R} \times \mathbb{R}^j$ *into* $\mathbb{R}^n$ *such that the relations*

$$\zeta(t) = \Pi x(t), \qquad x(t) = \phi(t, \zeta(t)), \tag{43}$$

provide a one-to-one correspondence between the amenable solutions $x(t)$ *of* (1) *and the solutions* $\zeta(t)$ *of the j-dimensional equation*

$$\frac{d\zeta}{dt} = \Pi f(t, \phi(t, \zeta)). \tag{44}$$

Proof of Corollary 8.1. If $(t, \zeta) \in \mathbb{R} \times \mathbb{R}^j$ then Theorem 8 shows that there exists a unique point $\phi(t, \zeta)$ in A_t such that $\zeta = \Pi\phi(t, \zeta)$. Since $A_t \subset \mathbb{R}^n$ this defines a function $\phi: \mathbb{R} \times \mathbb{R}^j \to \mathbb{R}^n$ which satisfies $x = \phi(t, \Pi x)$ for all x in A_t. Also (15) gives

$$\begin{aligned} 2^{1/2}|M||\zeta_1 - \zeta_2| &\geqslant |\phi(t, \zeta_1) - \phi(t, \zeta_2)| \\ &\geqslant |M^{-1}|^{-1}|\zeta_1 - \zeta_2| \qquad \text{for } \zeta_1, \zeta_2 \in \mathbb{R}^j. \end{aligned} \tag{45}$$

We now prove that $\phi(t, \zeta)$ is a continuous function of (t, ζ) at each point (r, ζ_0) in $\mathbb{R} \times \mathbb{R}^j$. Since $\phi(r, \zeta_0) \in A_r$, there exists an amenable solution $x_0(t)$ of (1) such that $x_0(r) = \phi(r, \zeta_0)$. Then $\zeta_0 = \Pi x_0(r)$. Also $x_0(t) = \phi(t, \Pi x_0(t))$ for all t because $x_0(t) \in A_t$. This and (45) give

$$\phi(t, \zeta) - \phi(r, \zeta_0) = [\phi(t, \zeta) - \phi(t, \Pi x_0(t))] + [x_0(t) - x_0(r)],$$

$$|\phi(t, \zeta) - \phi(r, \zeta_0)| \leqslant 2^{1/2}|M||\zeta - \Pi x_0(t)| + |x_0(t) - x_0(r)|.$$

Form (8), $|M^{-1}(x_0(r) - x_0(t))| \geqslant |\Pi(x_0(r) - x_0(t))| = |\zeta_0 - \Pi x_0(t)|$. Hence

$$|\phi(t, \zeta) - \phi(r, \zeta_0)| \leqslant 2^{1/2}|M||\zeta - \zeta_0| + (1 + 2^{1/2}|M||M^{-1}|)|x_0(t) - x_0(r)|.$$

This shows that $\phi(t, \zeta)$ is continuous at the point (r, ζ_0).

Since the linear mapping $\Pi: \mathbb{R}^n \to \mathbb{R}^j$ is independent of t we have $d(\Pi x(t))/dt = \Pi(dx/dt) = \Pi f(t, x(t))$, for every solution $x(t)$ of (1). When $x(t)$ is amenable, we have $x(t) = \phi(t, \Pi x(t))$ because $x(t) \in A_t$. Then $d(\Pi x(t))/dt = \Pi f(t, \phi(t, \Pi x(t))$ and $\zeta(t) = \Pi x(t)$ satisfies (43), (44). It only remains to show that every solution of (44) is of this form.

It is clear from (45) and (H1) that the right-hand side of (44) is continuous and locally Lipschitz in ζ at each point (r, ζ_0) of $\mathbb{R} \times \mathbb{R}^j$. By Picard's theorem there is only one solution $\zeta(t)$ of (44) such that $\zeta(r) = \zeta_0$. We showed above that there exists an amenable solution $x_0(t)$ of (1) such that $\Pi x_0(r) = \zeta_0$. Hence every solution $\zeta(t)$ of (44) is of the form $\Pi x(t)$, where $x(t)$ is an amenable solution of (1). This establishes Corollary 8.1.

These proofs of Theorem 8 and Corollary 8.1 have not assumed that (1) satisfies (H2). It was shown in Section 3 that if (1) does satisfy (H2) then $A_{r+\sigma} = A_r$ for all real r. This implies that $\phi(r + \sigma, \zeta) = \phi(r, \zeta)$ for all (r, ζ) in $\mathbb{R} \times \mathbb{R}^j$. Then the right-hand side of (44) is also σ-periodic in t. In particular, this is true when the hypotheses of Theorem 7 hold.

Proof of Theorem 7. Since (1) is assumed to have a solution $y(t)$ which is bounded in $[t_0, \infty)$ there exists a constant $\kappa \geqslant |y(t)|$ for all $t \geqslant t_0$. A sequence $\{m_\nu\}$ of positive integers can therefore be chosen such that $m_\nu \to \infty$ and $y(m_\nu \sigma) \to a$ as $\nu \to \infty$, where $a \in \mathbb{R}^n$. Suppose that the solution $x(t, a)$ with $x(0, a) = a$ exists throughout $r \leqslant t \leqslant s$ where $r < 0 < s$. Since (H2) holds $y(t + m_\nu \sigma)$ is a solution of (1) in $[t_0 - m_\nu \sigma, \infty)$ and therefore $y(t + m_\nu \sigma) \to x(t, a)$ pointwise in $[r, s]$ as $\nu \to \infty$. When ν is sufficiently large $\kappa \geqslant |y(t + m_\nu \sigma)|$ for $r \leqslant t \leqslant s$ and therefore $\kappa \geqslant |x(t, a)|$ for $r \leqslant t \leqslant s$. This shows that the point $x(t, a)$ in $\mathbb{R}^n$ can never meet the boundary ∂N of the ball $N = \{x \in \mathbb{R}^n : |x| \leqslant 1 + \kappa\}$. The solution $x(t, a)$ therefore exists and remains in N throughout $-\infty < t < \infty$. Hence (1) has a bounded amenable solution $x(t, a)$ and so satisfies all the hypotheses of Corollary 8.1.

Since $j = 2$, the σ-perdiodic differential equation (44) is two dimensional. All its solutions exist throughout $(-\infty, \infty)$ because (1) satisfies (H6). Since it has the bounded solution $\Pi x(t, a)$, Massera's second theorem shows that (44) has at least one σ-periodic solution $\zeta(t)$. Then Corollary 8.1 shows that $\phi(t, \zeta(t))$ is a σ-periodic solution of (1) because $\phi(t + \sigma, \zeta) = \phi(t, \zeta)$ for all (t, ζ) in $\mathbb{R} \times \mathbb{R}^j$. This establishes Theorem 7.

As a further illustration of the significance of Corollary 8.1 we now apply it to the study of uniformly almost periodic solutions of (1). Such a solution $x(t)$ has a generalised Fourier series of the form $\sum_{\nu=1}^{\infty} C_\nu \exp(i\Lambda_\nu t)$ and the countable set of numbers $\{\Lambda_\nu\}$ is called its *frequency spectrum*. A finite set of numbers $\kappa_1, \kappa_2, \ldots, \kappa_s$ is called a *rational base* of this frequency spectrum if every Fourier exponent Λ_ν can be expressed uniquely in the form $\Lambda_\nu = r_1 \kappa_1 + r_2 \kappa_2 + \cdots + r_s \kappa_s$ where $r_1, r_2, \ldots, r_s$ are rational numbers. If this is always true with integers $r_1, r_2, \ldots, r_s$ then $\kappa_1, \kappa_2, \ldots, \kappa_s$ is called an integral base of the frequency spectrum. Cartwright [2] considered equations (1) satisfying (H1), (H2) with $S = \mathbb{R}^n$ for which every solution $x(t)$ exists throughout $(-\infty, \infty)$. The *orbit closure* M of $x(t)$ is then defined to be the closure in $\mathbb{R}^n$ of the set of points $\{x(\nu\sigma)\}$ taken over all integers ν. Cartwright [2, p. 360] showed that if J is the topological dimension of the orbit closure M of a uniformly almost periodic solution $x(t)$ then $J \leqslant n - 1$ and the frequency spectrum of $x(t)$ has a rational base consisting of $J + 1$ elements. If, in addition, $J = n - 1$ then the frequency spectrum of $x(t)$ has an integral base of $J + 1$ elements and M is homeomorphic to a finite union of disjoint $(n-1)$-dimensional tori. For certain differential equations this result can be refined as follows:

THEOREM 9. *Suppose that* (1) *satisfies* (H1), (H2), (H3), (H4) *with* $S = \mathbb{R}^n$, $\lambda > 0$ *and* $j \geqslant 1$. *Suppose also that all solutions of* (1) *exist throughout* $-\infty < t < \infty$. *If* (1) *has a uniformly almost periodic solution* $u(t)$ *and* J *is the topological dimension of its orbit closure* M *then* $J \leqslant j - 1$ *and the frequency spectrum of* $u(t)$ *has a rational base consisting of* $J + 1$

elements. If, in addition, $J = j - 1$ then M is homeomorphic to a finite union of disjoint $(j-1)$-dimensional tori.

Proof of Theorem 9. Since the uniformly almost periodic solution $u(t)$ is bounded in $(-\infty, \infty)$ it is amenable and therefore (1) satisfies all the hypotheses of Corollary 8.1. Since (1) satisfies (H2) we have $\phi(t+\sigma, \zeta) = \phi(t, \zeta)$ in $\mathbb{R} \times \mathbb{R}^n$ and the right-hand side of (44) is σ-periodic in t. All solutions of (44) exist throughout $(-\infty, \infty)$, by Corollary 8.1. This verifies that (44) satisfies all the hypotheses of Cartwright 's theorem.

The orbit closure M of $u(t)$ is a compact invariant subset of $\mathbb{R}^n$. This ensures that if a solution $x(t)$ of (1) has $x(0) \in M$ then $x(t)$ is bounded in $(-\infty, \infty)$ and is therefore amenable. Hence, the amenable set $A_0 \supset M$. Then the mapping Π: $M \to \Pi M$ is homeomorphic and the sets M and ΠM have the same topological dimension J. Since ΠM is the orbit closure in $\mathbb{R}^j$ of the uniformly almost periodic solution $\Pi u(t)$ of (44), Cartwright's theorem shows that $J \leqslant j-1$. Also ΠM is homeomorphic to a finite union of disjoint $(j-1)$-dimensional tori in the case when $J = j-1$. Then the same is true of M because M and ΠM are homeomorphic. This establishes Theorem 9.

5. Verification of (H3), (H4)

To apply the theorems produced so far in this paper it is sufficient to know of the existence of a matrix P satisfying (H3), (H4) but the explicit computation of P is not required. In the present section we derive an inequality which provides a sufficient condition for the existence of P.

Consider the generalised feedback control equation

$$\frac{dx}{dt} = Ax + B\Phi(t, Cx), \tag{46}$$

in which $\Phi(t, y)$ is a continuous function from $\mathbb{R} \times \mathbb{R}^s$ into $\mathbb{R}^r$ and A, B, C are constant real matrices of types $n \times n$, $n \times r$, $s \times n$, respectively. When x varies over a set $S \subset \mathbb{R}^n$ the vector Cx varies over a subset of $\mathbb{R}^s$ which we denote by CS. We assume that there exists a positive constant $\Lambda(CS)$ such that

$$|\Phi(t, y_1) - \Phi(t, y_2)| \leqslant |y_1 - y_2| \, \Lambda(CS) \qquad \text{for } t \in \mathbb{R},\ y_1, y_2 \in CS. \tag{47}$$

The $r \times s$ matrix $\chi(z) = C(zI - A)^{-1}B$ is called the *transfer matrix* of (46). It is defined for all complex z with $\det(zI - A) \neq 0$. If this holds for all z with re $z = -\lambda$ we can define

$$\mu(\lambda) = \sup_{\omega \in \mathbb{R}} |\chi(i\omega - \lambda)|, \tag{48}$$

where $|K|$ denotes the spectral norm of an $r \times s$ matrix K. That is, $|K| = \sup(|Kv|/|v|)$ taken over all complex vectors $v \neq 0$ in $\mathbb{C}^s$. The following result will be proved:

THEOREM 10. *Suppose that* $\det(zI - A) \neq 0$ *for all* z *with* $\mathrm{re}\, z = -\lambda$. *If* (47) *holds and* $\Lambda(CS) < \mu(\lambda)^{-1}$ *then* (46) *satisfies* (H3) *for some constant symmetric matrix* P *and some constant* $\varepsilon > 0$. *Furthermore,* P *satisfies* (H4), *where* j *denotes the number of eigenvalues* ζ *of* A *which have* $\mathrm{re}\, \zeta > -\lambda$.

This theorem is closely related to results which are well known in stability theory (see [1, p. 211; 15]). The inequality $\Lambda(CS) < \mu(\lambda)^{-1}$ is only a sufficient condition for the existence of P and in many cases it is not best possible. However, Theorem 10 is of considerable generality because (1) can always been rewritten in the form (46) with arbitrary $n \times n$ matrices A, B, C, and

$$\Phi(t, y) = B^{-1}[f(t, C^{-1}y) - AC^{-1}y].$$

In practice, the matrices A, B, C can be chosen to exploit certain special properties which the function $f(t, x)$ may have. An advantage of this flexibility will be evident in Section 6.

Proof of Theorem 10. To verify (H3) it is sufficient to find a quadratic form $V(x)$ such that (5) holds for every pair of solutions $x_1(t)$, $x_2(t)$ of (46) in S. If $X(t) = x_1(t) - x_2(t)$ then (46) gives

$$\frac{dX}{dt} = AX + B[\Phi(t, Cx_1) - \Phi(t, Cx_2)] = AX + BQ(t)\, CX, \tag{49}$$

where $Q(t) = |CX|^{-2}[\Phi(t, Cx_1) - \Phi(t, Cx_2)](CX)^*$ when $CX \neq 0$ and $Q(t) = 0$ when $CX = 0$. In both cases (47) gives $|Q(t)| \leqslant \Lambda(CS)$ provided that $x_1(t), x_2(t) \in S$. If $Y(t) = e^{\lambda t}X(t)$ then X satisfies (49) if and only if

$$\frac{dY}{dt} = (A + \lambda I)\, Y + BQ(t)\, CY. \tag{50}$$

A result in stability theory [15, p. 206] shows that if $\Lambda(CS) < \mu(\lambda)^{-1}$ then there exist a constant $\varepsilon > 0$ and an Hermitian form $V_0(Y)$, which depend only on $A, B, C, \Lambda(CS), \mu(\lambda)$, such that $dV_0(Y)/dt \leqslant -\varepsilon|Y|^2$ for every (complex) solution $Y(t)$ of every equation of the form (50) which has $|Q(t)| \leqslant \Lambda(CS)$. For real solutions $Y(t)$ of (50) the hermitian form $V_0(Y) = Y^*P_0Y$ reduces to a real quadratic form $V(Y) = Y^*PY$, where $P = re P_0$. Since $Y = e^{\lambda t}X$ this gives $d[e^{2\lambda t}V(X)]/dt \leqslant -\varepsilon e^{2\lambda t}|X|^2$ and therefore (5) holds for every pair of solutions $x_1(t)$, $x_2(t)$ of (46). This proves that there exists P, ε such that (H3) holds when $f(t, x) = Ax + B\Phi(t, CX)$.

The linear equation $dY/dt = (A + \lambda I)\,Y$ is of the form (50) with $Q(t) \equiv 0$. Since this $Q(t)$ satisfies $|Q(t)| \leqslant \Lambda(CS)$ it follows that $dV_0(Y)/dt \leqslant -\varepsilon|Y|^2$ also holds for every complex solution of the linear equation. This then implies that (5) holds for every pair of solutions $x_1(t)$, $x_2(t)$ of the linear equation $dx/dt = Ax$. Hence P, ε also satisfy (H3) with $f(t, x) \equiv Ax$. In this case $f(t, x)$ has the Jacobian matrix $J(t, x) \equiv A$ and (32) shows that $(A + \lambda I)^* P + P(A + \lambda I)$ is negative definite. When this is so, [17, Lemma 1] asserts that the matrices $A + \lambda I$ and $-P$ have the same number of eigenvalues in the half plane $\operatorname{re} z > 0$ and no eigenvalues with $\operatorname{re} z = 0$. This proves that P satisfies (H4) where j denotes the number of eigenvalues of A in the half plane $\operatorname{re} z > -\lambda$. This establishes Theorem 10.

6. Circle Criterion for (H3), (H4)

Since (1) can be rewritten in the form (46) in many different ways, Theorem 10 provides many different conditions, each of which is sufficient for (1) to satisfy (H3), (H4) The aim of this section is to provide insight into this variety of conditions by restating Theorem 10 in a geometrical form. This will be done only for the special case of (46) when $r = s = 1$, because then the geometry is very simple.

If $r = s = 1$ then $\Phi(t, y)$ in (46) is a real-valued function of 2 real variables and $\chi(z) = C(zI - A)^{-1}B$ is a complex-valued function of z. For constant λ let N_λ denote the locus in the complex plane $\mathbb{C}$ of the point $\chi(i\omega - \lambda)^{-1}$ as ω varies over $\mathbb{R}$ If we suppose that $\det(zI - A) \neq 0$ when $\operatorname{re} z = -\lambda$, the identity

$$\det(zI - A - BkC) = (1 - \chi(z)\,k)\det(zI - A) \tag{51}$$

shows that $k \in N_\lambda$ if and only if the equation

$$0 = \det(zI - A - BkC) \tag{52}$$

has a root z with $\operatorname{re} z = -\lambda$.

For $k \in \mathbb{C}$ let $\nu(k)$ denote the number of roots of (52) in the halfplane $\operatorname{re} z > -\lambda$. As k varies in $\mathbb{C}$, the roots of (52) vary continuously with k and therefore $\nu(k)$ can change only when a root meets the line $\operatorname{re} z = -\lambda$. This happens only when k meets the curve N_λ. Thus, $\nu(k)$ remains constant when k varies over a connected component E of the open set $\mathbb{C} - N_\lambda$. Such a component E will be denoted by E^j_λ, where the integer j denotes the constant value of $\nu(k)$ for $k \in E$. On a diagram of the complex plane each component of $\mathbb{C}$-N_λ can be labelled with the appropriate symbol E^j_λ (see Fig. 1). The purpose of this labelling is illustrated by the following result:

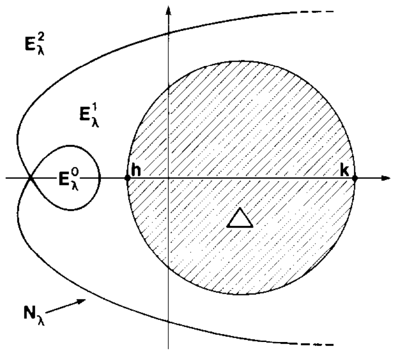

FIGURE 1

THEOREM 11. *Suppose that $r=s=1$ in* (46) *and that the set CS is an interval of $\mathbb{R}$ such that the partial derivative $\Phi_y(t, y)$ exists at all points of $\mathbb{R}\times CS$. Then* (46) *satisfies* (H3), (H4) *for some P, ε provided that there exists a closed circular disc Δ in the complex plane such that*

$$E_\lambda^j \supset \Delta \supset \operatorname*{Range}_{\mathbb{R}\times CS} \Phi_y(t, y). \tag{53}$$

This theorem is an analogue of the circle criterion for stability which is wellknown in control theory (see [1, pp. 220, 227]). In (53) Range $\Phi_y(t, y)$ denotes the set of all values taken by this real-valued function as (t, y) varies over the set $\mathbb{R}\times CS$. The requirement that CS be an interval is certainly satisfied if S is a connected subset of $\mathbb{R}^n$. The integer j in (H4) is determined by the symbol E_λ^j of the set containing Δ. The many different conditions which are sufficient for (46) to satisfy (H3), (H4) correspond in Theorem 11 to the many different circular discs Δ such that $E_\lambda^j \supset \Delta$.

Proof of Theorem 11. Let the circular disc Δ have centre k_1 and radius ρ. Then (53) is equivalent to the following:

$$\inf_{\omega\in\mathbb{R}} |\chi(i\omega-\lambda)^{-1}-k_1| > \rho \geqslant \sup_{\mathbb{R}\times CS} |\Phi_y(t, y)-k_1|. \tag{54}$$

Also, (53) shows that $k_1 \in E_\lambda^j$ and therefore the equation

$$0=\det(zI-A-Bk_1C) \tag{55}$$

has j roots z with re $z>-\lambda$ and none with re $z=\lambda$. Rewrite (46) as

$$\frac{dx}{dt}=A_1x+B\Psi(t, Cx), \tag{56}$$

where $A_1 = A + Bk_1C$ and $\Psi(t, y) = \Phi(t, y) - k_1 y$. The transfer function of (56) is $\chi_1(z) = C(zI - A_1)^{-1}B$. First, we prove that $\chi_1(z)^{-1} = \chi(z)^{-1} - k_1$, where $\chi(z) = C(z - A)^{-1}B$.

If u, v are column n-vectors with $v^*u \neq 1$ then

$$(I - uv^*)^{-1} = I + \beta uv^*, \qquad \text{where } \beta = (1 - v^*u)^{-1}.$$

This leads to the identity

$$v^*(I - uv^*)^{-1}u = (1 - v^*u)^{-1}v^*u.$$

With $u = (zI - A)^{-1}B$, $v^* = k_1C$, this gives $\chi_1(z) = (1 - k_1\chi(z))^{-1}\chi(z)$. Hence, $\chi_1(z)^{-1} = \chi(z)^{-1} - k_1$. This and (54) show that

$$\rho < \inf_{\omega \in \mathbb{R}} |\chi_1(i\omega - \lambda)^{-1}| = \mu_1(\lambda)^{-1},$$

where $\mu_1(\lambda) = \sup|\chi_1(i\omega - \lambda)|$ as ω varies over $\mathbb{R}$. This is the constant (48) appropriate to Eq. (56).

Since the roots of (55) are the eigenvalues of A_1, this matrix has j eigenvalues z with $\operatorname{re} z > -\lambda$ and none with $\operatorname{re} z = -\lambda$. Since CS is an interval and the right-hand side of (54) gives $\rho \geqslant |\Psi_y(t, y)|$ in $\mathbb{R} \times CS$, the mean-value theorem shows that

$$|\Psi(t, y_1) - \Psi(t, y_2)| \leqslant |y_1 - y_2|\,\rho, \qquad \text{for } t \in \mathbb{R},\ y_1, y_2 \in CS.$$

That is, $\Psi(t, y)$ satisfies (47) with $\Lambda(CS) = \rho$. Since $\rho < \mu_1(\lambda)^{-1}$, we can apply Theorem 10 to (56) to deduce that this equation satisfies (H3), (H4) for some P, ε. Then this is true of (46) because (56) is merely a rewritten version of (46). Hence Theorem 11 is proved.

A special case. Let $p(z)$, $q(z)$ be real polynomials of the form

$$p(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0, \qquad q(z) = q_m z^m + \cdots + q_1 z + q_0,$$

with $n > m$. Consider the real scalar differential equation

$$p(D)\,\xi = \Phi(t, q(D)\,\xi), \tag{57}$$

in which $D = d/dt$ and $\Phi(t, y)$ is a real function of 2 real variables. With $x = \operatorname{col}(\xi, D\xi, ..., D^{n-1}\xi)$, this reduces to

$$\frac{dx}{dt} = A_0 x + B_0 \Phi(t, C_0 x), \tag{58}$$

in which $B_0 = \operatorname{col}(0, 0, ..., 0, 1)$, $C_0 = (q_0, ..., q_m, 0, ..., 0)$ and A_0 is the $n \times n$

companion matrix of $p(z)$. If $Z=\operatorname{col}(1, z, z^2, ..., z^{n-1})$, it is easily verified that $(zI-A_0)Z=p(z)B_0$. Then

$$q(z)=C_0Z=C_0(zI-A_0)^{-1}p(z)B_0=p(z)\chi_0(z),$$

where $\chi_0(z)=C_0(zI-A_0)^{-1}B_0$ is the transfer function of (58).

Because of the simplicity of this expression $\chi_0(z)=q(z)/p(z)$ the corresponding curve N_λ in the complex plane can be easily sketched with the help of a pocket calculator. Theorems 2–11 can then be applied to (58) to gain information about (57). If, in particular, there exists a real constant l such that

$$y^{-1}\Phi(t, y)\to l, \qquad \text{uniformly in } -\infty<t<\infty \quad \text{as } |y|\to\infty, \tag{59}$$

then (58) satisfies (36) with $L=A_0+B_0lC_0$. Since (51) gives $\det(zI-L)=p(z)-lq(z)$, the result of Pliss [10, p. 42] shows that (58) is dissipative provided that $\operatorname{re} z<0$ for all roots of $p(z)-lq(z)=0$.

To illustrate the kind of explicit results to which this leads let us consider the simple example of the equation

$$\frac{d^4\xi}{dt^4}+4\frac{d^3\xi}{dt^3}+6\frac{d^2\xi}{dt^2}+4\frac{d\xi}{dt}=\Phi(t, \xi), \tag{60}$$

which is of the form (57) with $p(z)=(z+1)^4-1$, $q(z)\equiv 1$. In this case N_λ is the locus in $\mathbb{C}$ of the point $(i\omega+1-\lambda)^4-1$ as ω varies over $\mathbb{R}$. Figure 1 indicates its shape when $0<\lambda<1$. The function $\Phi(t, y)$ in (60) is assumed to satisfy

$$\Phi(t+\sigma, y)=\Phi(t, y), \qquad h\leqslant\Phi_y(t, y)\leqslant k \quad \text{for all } (t, y) \text{ in } \mathbb{R}^2, \tag{61}$$

where $[h, k]$ denotes the interval in which Δ intersects the real axis. Theorems 2, 6, 11 lead at once to the following convergence theorem:

THEOREM 12. *Suppose that $\Phi(t, y)$ satisfies* (61) *where h, k are any constants as shown in Fig. 1. Then any solution $\xi(t)$ of* (60) *converges to a σ-periodic solution as $t\to+\infty$ provided that $\xi(t)$, $\xi'(t)$, $\xi''(t)$, $\xi'''(t)$ are all bounded in some interval $[t_0, \infty)$. If, in addition,* (59) *holds with $-5<l<0$ then every solution $\xi(t)$ of* (60) *converges to a periodic solution as $t\to+\infty$ and at least one periodic solution is Lyapunov stable. If, furthermore, $\Phi(t, y)$ is analytic in $\mathbb{R}^2$ then* (60) *has only a finite number of periodic solutions and at least one of these is asymptotically stable.*

With $\lambda=\frac{1}{2}$, a pocket calculator shows that $h=-0.9$, $k=21$ is one pair of constants suitable for Theorem 12.

To make a similar application of Theorem 7, the disc Δ in Fig. 1 must be shifted to the left-hand side of N_λ so that $\Delta \subset E_\lambda^2$. In this case it is convenient to choose $\lambda = 1$ when the curve N_λ lies wholly on the segment $[-1, \infty)$ of the real axis. Then $\Delta \subset E_\lambda^2$ holds if $h < k < -1$. Theorems 7, 9, 11 lead at once to the following analogue of Massera's second theorem:

THEOREM 13. *Suppose that $\Phi(t, y)$ satisfies* (61), *where h, k are any constants such that $h < k < -1$. If* (60) *has a solution $\xi(t)$ such that $\xi(t)$, $\xi'(t)$, $\xi''(t)$, $\xi'''(t)$ are all bounded in some interval $[t_0, \infty)$ then* (60) *has at least one σ-periodic solution. Furthermore, the frequency spectrum of any uniformly almost periodic solution of* (60) *has a rational base of at most* 2 *elements.*

In the special case $\Phi(t, y) = -5y + \cos y - \cos \sin t$, the hypotheses of Theorem 13 hold with $\sigma = \pi$, $h = -6$, $k = -4$. In this case (60) has the subharmonic solution $\xi = \sin t$ from which Theorem 13 is able to predict the existence of at least one π-periodic solution.

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